PHYSICAL REVIEW E **72**, 046502 (2005)

Formulation of nonlinear chromaticity in circular accelerators by canonical perturbation method

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(Received 29 June 2005; published 14 October 2005)

The formulation of nonlinear chromaticity in circular accelerators based on the canonical perturbation method is presented. Since the canonical perturbation method directly relates the tune shift to the perturbation Hamiltonian, it greatly simplifies the calculation of the nonlinear chromaticity. The obtained integral representation for nonlinear chromaticity can be systematically extended to higher orders.

DOI: 10.1103/PhysRevE.72.046502 PACS number(s): 41.85.−p

I. INTRODUCTION

For designing and operating circular accelerators it is important to precisely understand the variations of lattice parameters by the perturbations of focusing forces. For storage rings with strong focusing forces a linear approximation is, however, not sufficient for this demand. In addition, modern storage rings dedicated to a high brilliant synchrotron radiation source require wide momentum acceptance to achieve small beam loss and long beam lifetime. Then for the purpose of dealing with the particle motion with far off-momentum, we need to know the higher order terms of nonlinear chromaticity. In the preceding paper [1] we hence derived the rigorous formula for higher order chromaticity up to the third. The effectiveness of the formula was confirmed by comparison with the experiments at the SPring-8 storage ring.

In the derivation of the formula we used the transfer matrix method. In order to extract the higher order chromaticity in the closed form, we had to use the Fourier transform of the higher order modulation of focusing force. Although the calculation is somewhat simplified by using the Hill determinant as pointed out by Wang [2], the formulation is still complicated due to the unavoidable use of the Fourier transform. It is known that the canonical perturbation method is superior to other methods in perturbation calculations. Hence in this paper we apply the canonical perturbation method to the calculation of the nonlinear chromaticity. The integral representations of the nonlinear chromaticity takes a simpler form than those derived in the previous paper, which can be systematically extended to higher orders.

Before applying the canonical perturbation method to the formulation of the nonlinear chromaticity, we briefly describe the general canonical perturbation theory [3–5] in Sec. II as preliminary. In Sec. III we then derive the higher order formula of nonlinear chromaticity by means of the canonical perturbation method and show the equivalence with the previous ones obtained through the transfer matrix.

II. CANONICAL PERTURBATION THEORY

We first review the canonical perturbation method. We consider the problem described by the Hamiltonian

$$H = H_0(J,s) + V(\phi,J,s), \tag{1}$$

where H is written in terms of action-angle variables of unperturbed problem described by the unperturbed Hamiltonian H_0, J and ϕ are respectively the action and angle variables, V is a perturbation potential, and s is the independent variable, i.e., the path length of a circular accelerator in the present case. For the sake of brevity we restrict our discussion to the one-dimensional case while it is easy to extend it to the arbitrary dimension. It is emphasized that in the Hamilton-Jacobi formalism the unperturbed Hamiltonian H_0 does not depend on the angle variable ϕ. The unperturbed Hamiltonian H_0 describing the betatron motion in a circular accelerator is given by [3–5]

$$H_0(J,s) = \frac{J}{\beta(s)} \tag{2}$$

with the betatron function $\beta(s)$. The equations of motion of the unperturbed system described by the Hamiltonian H_0 are

$$\frac{dJ}{ds} = -\frac{\partial H_0}{\partial \phi} = 0, \tag{3}$$

$$\frac{d\phi}{ds} = \frac{\partial H_0}{\partial J} = \frac{1}{\beta(s)}, \tag{4}$$

and the solution is

$$J = \text{const}, \tag{5}$$

$$\phi(s) = \phi(0) + \int_0^s \frac{ds'}{\beta(s')}. \tag{6}$$

In the following we suppose that a perturbation potential V can be expanded in power series in measure δ identifying the perturbation as

$$V(\phi,J,s) = \sum_{n=1} \delta^n V_n(\phi,J,s). \tag{7}$$

In the canonical perturbation theory, a problem is solved by obtaining the action-angle variables in the perturbed system by means of the canonical transformation. Here we consider the canonical transformation generated by δ-dependent type 2 generating function $S(\phi,\bar{J},s)$ where $\bar{J}$ is the new action

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*Electronic address: takao@spring8.or.jp variable. Since canonical transformation S is close to identity transformation provided V is small, S is expanded in a power series in δ as

$$S(\phi,\bar{J},s) = \phi\bar{J} + \sum_{n=1} \delta^n S_n(\phi,\bar{J},s). \tag{8}$$

The new coordinates and the Hamiltonian are then given by

$$\bar{\phi} = \phi + \sum_{n=1} \delta^n S_{n,\bar{J}}(\phi,\bar{J},s), \tag{9}$$

$$J = \bar{J} + \sum_{n=1} \delta^n S_{n,\phi}(\phi,\bar{J},s), \tag{10}$$

$$\bar{H} = H + \sum_{n=1} \delta^n S_{n,s}(\phi,\bar{J},s), \tag{11}$$

where the subscripts after commas indicate the partial differentiations with respect to the variables.

If ϕ was known, the perturbed Hamiltonian can be written as a function of $\bar{J}$ and s alone. Inserting the action J given by Eq. (10) into the Hamiltonian $\bar{H}$ given by (11) with Eq. (1), we have

$$\bar{H}(\bar{J},s) = H_0(\bar{J},s) + \sum_{n=1} \delta^n K_n(\bar{J},s), \tag{12}$$

where

$$K_1 = V_1 + H_{0,\bar{J}}S_{1,\phi} + S_{1,s}, \tag{13}$$

$$K_2 = V_2 + V_{1,\bar{J}}S_{1,\phi} + H_{0,\bar{J}}S_{2,\phi} + S_{2,s}, \tag{14}$$

$$K_3 = V_3 + V_{2,\bar{J}}S_{1,\phi} + V_{1,\bar{J}}S_{2,\phi} + \tfrac{1}{2}V_{1,\bar{J}\bar{J}}S_{1,\phi}^2 + H_{0,\bar{J}}S_{3,\phi} + S_{3,s},$$

$$\cdots. \tag{15}$$

If the new variables are canonical, the perturbed Hamiltonian, and hence the higher order terms K_n, does not contain the angle variable. The generating functions S_n are determined so as to eliminate the angle-variable-dependent part of K_n.

Note that in the perturbation Hamiltonian K_n the highest order component of generating function S_n appears in the form of $S_{n,s}+H_{0,\bar{J}}S_{n,\phi}$. Excluding the highest order term $S_{n,s}+H_{0,\bar{J}}S_{n,\phi}$ from K_n and defining the rest as $F_n(V_1,\ldots,V_n,S_1,\ldots,S_{n-1})$, we have

$$K_n = F_n + S_{n,s} + H_{0,\bar{J}}S_{n,\phi}. \tag{16}$$

Using the canonical invariance of the action variables, we can conclude that the generating function S_n should be periodic in the angle variable ϕ. Then we find that the average of the derivative $S_{n,\phi}$ over the angle variable ϕ vanishes

$$\langle S_{n,\phi}\rangle \equiv \frac{1}{2\pi}\oint d\phi S_{n,\phi} = 0 \tag{17}$$

as well as $\langle S_{n,s}\rangle=0$. Here the symbol $\langle\cdot\rangle$ denotes the average over the angle variable ϕ. Since H_0 is constant in ϕ, we find

$$\langle S_{n,s} + H_{0,\bar{J}}S_{n,\phi}\rangle \equiv 0. \tag{18}$$

Then the perturbation Hamiltonian K_n, not including the angle variable ϕ obtained by averaging Eq. (16), yields

$$K_n = \langle F_n\rangle(V_1,\ldots,V_n,S_1,\ldots,S_{n-1}). \tag{19}$$

Defining the oscillating part of F_n in ϕ as

$$\{F_n\} = F_n - \langle F_n\rangle, \tag{20}$$

we have the determining equation for S_n

$$S_{n,s} + H_{0,\bar{J}}S_{n,\phi} = -\{F_n\}(V_1,\ldots,V_n,S_1,\ldots,S_{n-1}).$$

We thus obtain the recurrent equation for the generating function S_n and can solve it order by order to find the perturbation Hamiltonian K_n.

For the betatron motion $H_{0,\bar{J}}=1/\beta(s)$ a recurrence equation is written as

$$S_{n,s}(\phi,\bar{J},s) + \frac{1}{\beta(s)}S_{n,\phi}(\phi,\bar{J},s) = -\{F_n\}(\phi,\bar{J},s).$$

Since both S_n and F_n are periodic functions of ϕ, they can be expanded in Fourier series,

$$S_n(\phi,\bar{J},s) = \sum_m \tilde{S}_n(m,\bar{J},s)e^{im\phi}, \tag{21}$$

$$\{F_n\}(\phi,\bar{J},s) = \sum_m \tilde{F}_n(m,\bar{J},s)e^{im\phi}. \tag{22}$$

Then the equation to be solved for S_n becomes

$$\frac{\partial\tilde{S}_n}{\partial s}(s) + \frac{im}{\beta(s)}\tilde{S}_n(s) = -\tilde{F}_n(s), \tag{23}$$

which has the periodic solution in s

$$\tilde{S}_n(m,\bar{J},s) = \frac{i}{2\sin(m\pi\nu)} \times \int_s^{s+C} ds'\, e^{im[\psi(s')-\psi(s)-\pi\nu]}\tilde{F}_n(m,\bar{J},s').$$

Here C is the circumference of a circular accelerator, and we define the unperturbed betatron phase $\psi(s)=\int_0^s ds'/\beta(s')$, and the tune $\nu=\int_0^C ds'/[2\pi\beta(s')]$. Finally, the full expression for S_n is given by

$$S_n(\phi,\bar{J},s) = \sum_m \frac{i}{2\sin(m\pi\nu)} \times \int_s^{s+C} ds'\, e^{im[\phi+\psi(s')-\psi(s)-\pi\nu]}\tilde{F}_n(m,\bar{J},s'). \tag{24}$$

If we know the explicit form of the perturbation potential V, we can solve the problem.

III. APPLICATION TO NONLINEAR CHROMATICITY

Now we apply the canonical perturbation method to nonlinear chromaticity. In this case the relative momentum deviation $\delta=(p-p_0)/p_0$, with p_0 being the nominal momentum, corresponds to the measure of the perturbation. The perturbation potential V is given by

$$V_n(\phi,J,s)=\tfrac{1}{2}J\beta(s)G_n(s)(1+\cos 2\phi), \tag{25}$$

where G_n is the perturbation of the focusing forces. The explicit representations of G_n in terms of the higher order dispersion functions up to the third are given in [1]. Since V_n is linear in the action variable $\bar{J}$, we have, e.g., $V_{n,\bar{J}\bar{J}}=0$ and find

$$F_n=V_n+\sum_{m=1}^{n-1}V_{n-m,\bar{J}}S_{m,\phi}. \tag{26}$$

Since the perturbation potential V_n consists of a constant and the second harmonics in ϕ only as found in Eq. (25), the same harmonics of the generating function S_n can contribute to the perturbation Hamiltonian $K_n=\langle F_n\rangle$ due to the average over ϕ.

The equation of motion for the angle variable in the perturbed system is

$$\frac{d\bar{\phi}}{ds}=\frac{\partial\bar{H}}{\partial\bar{J}}, \tag{27}$$

whose integration over the circumference yields the tune. Hence the nth-order chromaticity ξ_n, which is the tune shift due to the higher order perturbation Hamiltonian K_n, is represented as

$$\xi_n=\frac{1}{2\pi}\int_s^{s+C}ds'\frac{\partial K_n}{\partial\bar{J}}(\bar{J},s'). \tag{28}$$

For example, in the case of the first-order perturbation,

$$K_1(\bar{J},s)=\tfrac{1}{2}\bar{J}\beta(s)G_1(s), \tag{29}$$

and we immediately obtain

$$\xi_1=\frac{1}{4\pi}\int_s^{s+C}ds'\beta(s')G_1(s'), \tag{30}$$

which is certainly the linear chromaticity as proven in [1].

To proceed to the next order, we need the first-order generating function S_1. Since the periodic part $\{F_1\}$ is

$$\{F_1\}(\phi,\bar{J},s)=\tfrac{1}{2}\bar{J}\beta(s)G_1(s)\cos 2\phi, \tag{31}$$

the first-order generating function S_1 can be easily derived from Eq. (24):

$$\begin{aligned}S_1(\phi,\bar{J},s)=&-\frac{\bar{J}}{4\sin(2\pi\nu)}\int_s^{s+C}ds'\beta(s')G_1(s')\\&\times\sin 2[\phi+\psi(s')-\psi(s)-\pi\nu].\end{aligned} \tag{32}$$

Inserting the above representation of the generating function S_1 together with those of V_1 and V_2 into the second-order perturbation Hamiltonian K_2,

$$K_2(\bar{J},s)=\langle V_2\rangle+\langle V_{1,\bar{J}}S_{1,\phi}\rangle, \tag{33}$$

we obtain the second-order chromaticity

$$\begin{aligned}\xi_2=\frac{1}{4\pi}\Bigg(&\int_s^{s+C}ds'\beta(s')G_2(s')\\&-\frac{1}{4\sin(2\pi\nu)}\int_s^{s+C}ds'\beta(s')G_1(s')\int_{s'}^{s'+C}ds''\beta(s'')G_1(s'')\\&\times\cos 2[\psi(s'')-\psi(s')-\pi\nu]\Bigg).\end{aligned} \tag{34}$$

This integral representation differs from that derived in [1] by the integration region in the twofold integral. The equivalence between the two integral representations can be proven by simply changing integration variables.

Although the labor to derive the higher order formula for nonlinear chromaticity increases exponentially order by order, the canonical perturbation procedure can be continued systematically. While it is possible to give the general representation for an arbitrary order chromaticity, we limit ourselves to the third-order one, which is sufficient for practical use as shown in [1]. Since $F_2=V_2+V_{1,J}S_{1,\phi}$, the second-order generating function S_2 is given by

$$\begin{aligned}S_2(\phi,\bar{J},s)=&-\frac{\bar{J}}{4\sin(2\pi\nu)}\int_s^{s+C}ds'\beta(s')G_2(s')\sin 2[\phi+\psi(s')-\psi(s)-\pi\nu]\\&+\frac{\bar{J}}{8\sin^2(2\pi\nu)}\int_s^{s+C}ds'\beta(s')G_1(s')\int_{s'}^{s'+C}ds''\beta(s'')G_1(s'')\sin 2[\phi+\psi(s'')-\psi(s)-2\pi\nu]\\&+\frac{\bar{J}}{8\sin(2\pi\nu)\sin(4\pi\nu)}\int_s^{s+C}ds'\beta(s')G_1(s')\int_{s'}^{s'+C}ds''\beta(s'')G_1(s'')\sin 2[2\phi+\psi(s')+\psi(s'')-2\psi(s)-3\pi\nu].\end{aligned}$$

Note that the fourth harmonic in ϕ appears in the above equation as a result of the product of the perturbation potential. In general, as the order of perturbation becomes high, the higher harmonics appear and make the calculations complicated. Fortunately, since the perturbation potential V_n possesses the second harmonic 2ϕ only, the harmonic component of S_n survives after being multiplied with V_n and averaging over ϕ. In other words, the second harmonic component of the generating function S_n can contribute to the higher order chromaticity.

Since we already know the explicit forms of the generating functions S_1 and S_2, we can derive the integral representation of the third-order chromaticity:

$$\begin{aligned}\xi_3 = \frac{1}{4\pi}\Bigg(& \int_s^{s+C} ds'\beta(s')G_3(s') - \frac{1}{4\sin(2\pi\nu)}\int_s^{s+C} ds'\beta(s')G_2(s')\int_{s'}^{s'+C} ds''\beta(s'')G_1(s'')\cos 2\{\psi(s'') - \psi(s') - \pi\nu\} \\ & - \frac{1}{4\sin(2\pi\nu)}\int_s^{s+C} ds'\beta(s')G_1(s')\int_{s'}^{s'+C} ds''\beta(s'')G_2(s'')\cos 2\{\psi(s'') - \psi(s') - \pi\nu\} \\ & + \frac{1}{8\sin^2(2\pi\nu)}\int_s^{s+C} ds'\beta(s')G_1(s')\int_{s'}^{s'+C} ds''\beta(s'')G_1(s'') \times \int_{s''}^{s''+C} ds'''\beta(s''')G_1(s''')\cos 2[\psi(s''') - \psi(s') - 2\pi\nu]\Bigg).\end{aligned} \tag{35}$$

Again the representations of the multiple integrals derived by the present canonical formulation differs from those by the previous transfer matrix method [1]. The equivalence of both the representations can be easily proven by the help of the Fourier transformation used in [1]. In the numerical calculation of the higher order chromaticity the direct integration of the above representation is more convenient than using the Fourier series.

IV. CONCLUDING REMARKS

In a preceding paper [1] we presented the rigorous formula for nonlinear chromaticity and confirmed the effectiveness by comparing the numerical estimations with the experiments at the SPring-8 storage ring. Here we show that the canonical perturbation method greatly improves the derivation of the higher order formula for nonlinear chromaticity. The integral representation of higher order chromaticity possesses the simple form compared to that derived by the transfer matrix method. Hence the canonical perturbation formulation of nonlinear chromaticity can be extended to higher order straightforwardly.

ACKNOWLEDGMENTS

The author would like to thank Drs. H. Tanaka, K. Soutome, and J. Schimizu for the collaboration.

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